TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 1045

THE HEAT TRANSFER TO A PLATE IN FLOW AT HIGH SPEED

By E. Eckert and O. Drewitz

Forschung, Vol. 11, No. 3, May-June 1940

Washington
May 1943

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 1045

THE HEAT TRANSFER TO A PLATE IN FLOW AT HIGH SPEED

By E. Eckert and O. Drewitz

SUMMARY

The heat transfer in the laminar boundary layer of a heated plate in flow at high speed can be obtained by integration of the conventional differential equations of the boundary layer, so long as the material values can be regarded as constant. This premise is fairly well satisfied at speeds up to about twice the sonic speed and at not excessive temperature rise of the heated plate. The general solution of the equation includes Pohlhausen's specific cases of heat transfer to a plate at low speeds and of the plate thermometer. The solution shows that the heat transfer coefficient at high speed must be computed with the same equation as at low speed, when it is referred to the difference of the wall temperature of the heated plate in respect to its "natural temperature." Since this fact follows from the linear structure of the differential equation describing the temperature field, it is equally applicable to the heat transfer in the turbulent boundary layer.

INTRODUCTION

The development of skin radiators and the utilization of heat against icing, together with the necessity of sealing the pressure cabins of stratosphere aircraft against low outside temperature, have introduced increasing interest in heat transfer problems. English tests, as well as those by Seibert (reference 1) have indicated that the heat transfer to wing profiles can be quite accurately predicted from that of the flow along a flat plate. Even the heat transfer to an airplane fuselage can be simplified to that along the flat plate. Our knowledge on heat transfer past a plate extends, it is true, only to low airspeeds.

*"Der Wärmeübergang an eine mit grosser Geschwindigkeit längs angeströmte Platte. Forschung, vol. 11, no. 3, May-June, 1940, pp. 116-124.

And the cited measurements on wing profiles themselves were made only at airspeeds from 20 to 60 meters per second. An application of the test data to speeds reached by modern high-speed aircraft involves two difficulties. First, the compressibility of air must be considered at such speeds; second, the heat introduced in the boundary layer by internal friction reaches values far from negligible. The effect of these phenomena on the heat transfer to the flat plate will be analyzed in the present report.

Heat transfer in the laminar boundary layer.- Starting with the heat transfer in the boundary layer to a plate in longitudinal flow, the plate is regarded as infinite perpendicular to the direction of flow. Fixing, in this flow, a coordinate system with the x axis in the plane of the plate placed in stream direction and the y axis perpendicular to the plate, the differential equations of the boundary layer read:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{1}$$

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial y}\right)^2 = g\ c_p\ \rho\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) \tag{3}$$

where

u velocity component in x direction

v velocity component in y direction

T temperature

ρ air density

η viscosity

λ thermal conductance

c_p specific heat per unit weight at constant pressure

g gravitational acceleration

The equations are expressed in the engineering system of units, g cancels out in the physical system, because in the latter the specific heat is referred to unit mass.* Outside of the boundary layer the speed is to have the constant magnitude u_0; at the plate itself it is zero. Prandtl's hypothesis of small boundary layer thickness which leads to the above differential equations, shows that the pressure in the boundary layer of the flat plate and hence in the entire field of flow is constant. The material quantities, therefore, depend only on the temperature, that is, for air the density varies, according to the gas equation, inversely proportional to the absolute temperature, the viscosity and the thermal conductance about proportional to the 0.75 power of the absolute temperature, while the specific heat remains practically unchanged. The greatest temperature differences that can occur as a result of the interanl friction in air (and in any substance where Prandtl number is less than unity), are of the order of the adiabatic temperature rise $u_0^2/2 g c_p$. At sonic speed expressed with $a_s = \sqrt{g\ (\kappa - 1)\ c_p\ T}$, the adiabatic temperature rise is $\Delta T = \frac{a_s^2}{2\ g\ c_p} = \frac{\kappa - 1}{2}\ T$ and hence the relative variation of density in the boundary layer due to the temperature accumulation $\frac{\Delta\rho}{\rho} = \frac{\Delta T}{T} = \frac{\kappa - 1}{2}$ for air with an adiabatic exponent $\kappa = 1.4$, is therefore, $\frac{\Delta\rho}{\rho} = 20$ percent. At **air**speeds up to velocity of sound the material values therefore do not vary very much, hence may be regarded as constant, provided that the impressed temperature difference of the plate relative to gas in undisturbed flow is not excessive. Bussmann and Von Karman** have solved the friction and temperature conditions in the laminar boundary layer of a plate for the specific case of flowing gas having the Prandtl number $Pr = 1$, where the variation of the material values is considered. The calculations indicate in agreement with the previous estimation

*In equation (3) and those following, the mechanical equivalent of heat A is omitted, since it is superfluous by dimensionless presentation of the formulas; it merely requires that the equation 1 kcal = 427 mkg is applicable.

**See references 2 and 3.

that the effect of the variable material values does not become perceptible except at great Mach numbers (greater than 2). Limited to Mach numbers smaller or only a little greater than unity, the differential equations (1) to (3) read

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1')$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2')$$

$$\varepsilon\, c_p\, \rho\left(u\frac{\partial \vartheta}{\partial x} + v\frac{\partial \vartheta}{\partial y}\right) = \lambda\frac{\partial^2 \vartheta}{\partial y^2} + \nu\,\rho\left(\frac{\partial u}{\partial y}\right)^2 \qquad (3')$$

where

$\nu = \eta/\rho$ kinematic viscosity

ϑ . signifies the temperature increase relative to the value in undisturbed air stream

On introducing the stream function $\psi\left(u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}\right)$

equation (1') is fulfilled. Equation (2') reduces, according to Blasius (reference 4) with the variables $\xi = \frac{1}{2}\, y\sqrt{\frac{u_0}{\nu x}}$ and $\zeta = \frac{\psi}{\sqrt{\nu\, u_0\, x}}$, to the conventional differential equation

$$\frac{d^3\zeta}{d\xi^3} + \zeta\frac{d^2\zeta}{d\xi^2} = 0 \qquad (4)$$

for which the limiting conditions are: $u_{\xi=0}=0$ and $u_{\xi=\infty}=u_0$. Since, $u = \frac{u_0}{2}\frac{d\zeta}{d\xi}$ according to the above, $\left(\frac{d\zeta}{d\xi}\right)_{\xi=0}=0$ and $\left(\frac{d\zeta}{d\xi}\right)_{\xi=\infty} = 2$. This solution was given by Blasius.

Introducing the new variables and inserting the Prandtl number $Pr = \eta\ c_p\ g/\lambda$, equation (3') transforms, according to Pohlhausen (reference 5), to

$$\frac{d^2\vartheta}{d\xi^2} + Pr\,\zeta\,\frac{d\vartheta}{d\xi} = -\frac{Pr}{2}\,\frac{u_0^{\,2}}{2gc_p}\left(\frac{d^2\zeta}{d\xi^2}\right)^2 = f(\xi) \tag{5}$$

This is a linear inhomageneous differential equation for solving the increase of temperature ϑ. In it ζ is a function of ξ known through the solution of equation (4), the other quantities are constant. By introducing a new unknown for $d\vartheta/d\xi$ it can be reduced to a differential equation of the first order.

The solution of the related homogeneous differential equation

$$\frac{d^2\vartheta_1}{d\xi^2} + Pr\,\zeta\,\frac{d\vartheta_1}{d\xi} = 0 \tag{6}$$

shows after integration

$$\vartheta_1(\xi) = C_1 + C_2\int_0^\xi e^{-Pr\int_0^\xi \zeta d\xi}\,d\xi \tag{7}$$

Physically, the posting of the disturbing term $f(\xi)$ equal to zero in equation (5) signifies that the heat introduced by internal friction is neglected in the solution of the temperature field. This is admissible at low speeds, because the speed u_0 enters squared in $f(\xi)$.

On these premises temperature differences in the flow are formed only when the plate is heated and dissipates heat on the gas and conversely withdraws heat from the gas by cooling. With T_w signifying the plate temperature and T_0 the gas temperature in undisturbed flow, the limiting conditions l.c., to which the solution of the differential equation must be fitted, read:

$$l.c.1.\ \vartheta_1(0) = T_w - T_0; \qquad l.c.2.\ \vartheta_1(\infty) = 0$$

The constants C_1 and C_2 are obtained from the general solution (7) by entering the limiting conditions. Thus

$$\vartheta_1(\xi) = (T_w - T_o)\left[1 - a\int_0^\xi e^{-Pr\int_0^\xi \zeta d\xi} d\xi\right] \tag{8}$$

in which $\frac{1}{a} = \int_0^\infty e^{-Pr\int_0^\xi \zeta d\xi} d\xi$ - that is, a function of the Prandtl number.

This solution is identical with that given in a slightly different form by Pohlhausen (reference 5).

Equation (8) is evaluated in figure 8; it indicates the temperature field for different Pr at small flow velocities. At the wall, $u = 0$ and $v = 0$; hence $\frac{\partial^2 \vartheta_1}{\partial y^2} = 0$, according to equation (3'). The curves begin with infinitely small curvature at $\xi = 0$; they then deflect quite sharply in the horizontal asymptote. The field of velocity itself is readily apparent. Substituting the unknown $\vartheta' = 2\left(1 - \frac{\vartheta_1}{T_w - T_o}\right)$ for ϑ_1 in equation (6) does not alter the form of the equation, but the limiting conditions for ϑ' read: $\vartheta'(0) = 0$ and $\vartheta'(\infty) = 2$. Then the same differential equation (4) or (6) and the same limiting conditions are applicable to $d\zeta/d\xi$ and ϑ' at $Pr = 1$. Both functions of ξ must therefore be identical: $\frac{d\zeta}{d\xi} = \vartheta'$. It follows that $1 - \frac{\vartheta_1}{T_w - T_o} = \frac{u}{u_o}$. The lines in figure 1 for $Pr = 1$ therefore indicate, at the same time, the velocity profile for all Pr. The thickness ratio of the frictional boundary layer to the thermal boundary layer is therefore also immediately obtainable from the graph. For an oil with a $Pr = 1000$, for instance, the line $Pr = 1000$ indicates the extent of the thermic boundary layer; the line $Pr = 1$, the extent of the frictional boundary layer. The latter extends for this oil ten times as far into the fluid as the former.

The amount of heat given off by the plate per unit area per unit time at point x is

$$q = -\lambda \left(\frac{\partial \vartheta_1}{\partial y}\right)_{y=0} = -\frac{\lambda}{2}\sqrt{\frac{u_0}{\nu x}}\left(\frac{d\vartheta_1}{d\xi}\right)_{\xi=0}$$

$$= \frac{\lambda}{2}\sqrt{\frac{u_0}{\nu x}}\ a\ (T_w - T_o) = \alpha(T_w - T_o)$$

where α is the coefficient of heat transfer. Then the Nusselt number is

$$Nu = \frac{\alpha x}{\lambda} = \frac{a}{2}\sqrt{\frac{u_0 x}{\nu}} = \frac{a}{2}\sqrt{Re} \qquad (9)$$

Re is the Reynolds number formed with the distance from the plate front edge. According to Pohlhausen (reference 5) $a = 0.664 \sqrt[3]{Pr}$ is a good approximation, which holds up to $Pr = 1000$ according to Ten Bosch (reference 6). The formula for the heat transfer coefficient herewith becomes

$$Nu = 0.332 \sqrt{Re} \sqrt[3]{Pr} \qquad (10)$$

Integration with respect to the plate length affords the total heat removal from start of plate to point x

$$Q = \int_0^x q\ dx = \lambda \sqrt{\frac{u_0 x}{\nu}}\ a\ (T_w - T_o)$$

and the Nusselt number formed with the mean heat transfer coefficient from beginning of plate to point x

$$Nu_m = \frac{\alpha_m x}{\lambda} = a\sqrt{Re} = 0.664 \sqrt{Re} \sqrt[3]{Pr} \qquad (11)$$

<u>The heat transfer in the laminar boundary layer at high speed</u>.- The general solution of the inhomogeneous differential equation (5) is obtained with the solution (equation (7)) of the homogeneous equation by "variation of the constants." It reads:

$$\vartheta(\xi) = C_1' + C_2' \int_0^{\xi} e^{-Pr\int_0^{\xi} \zeta d\xi} d\xi$$

$$- \frac{u_0^2}{2\, g\, c_p} \frac{Pr}{2} \int_0^{\xi} \left[e^{-Pr\int_0^{\xi} \zeta d\xi} \int_0^{\xi} \left(\frac{d^2\zeta}{d\xi^2} \right)^2 e^{Pr\int_0^{\xi} \zeta d\xi} d\xi \right] d\xi \qquad (12)$$

The temperature field at heat transfer to the plate with great flow velocities follows by appropriate choice of limiting conditions. Given the wall temperature T_w and the temperature in undisturbed flow T_0, they read:

$$l.c.1 \quad \vartheta(0) = T_w - T_0; \qquad l.c.2 \quad \vartheta(\infty) = 0$$

The related solution reads:

$$\vartheta(\xi) = (T_w - T_0) \left[1 - a \int_0^{\xi} e^{-Pr\int_0^{\xi} \zeta d\xi} d\xi \right]$$

$$+ \frac{u_0^2}{2\, g\, c_p} \left(a\, b \int_0^{\xi} e^{-Pr\int_0^{\xi} \zeta d\xi} d\xi \right.$$

$$\left. - \frac{Pr}{2} \int_0^{\xi} \left[e^{-Pr\int_0^{\xi} \zeta d\xi} \int_0^{\xi} \left(\frac{d^2\zeta}{d\xi^2} \right)^2 e^{Pr\int_0^{\xi} \zeta d\xi} d\xi \right] d\xi \right) \qquad (13)$$

where

$$b = \frac{Pr}{2} \int_0^{\infty} \left[e^{-Pr\int_0^{\xi} \zeta d\xi} \int_0^{\xi} \left(\frac{d^2\zeta}{d\xi^2} \right)^2 e^{Pr\int_0^{\xi} \zeta d\xi} d\xi \right] d\xi$$

The heat removal of the plate at point x per unit surface is

$$q = -\lambda \left(\frac{\partial \vartheta}{\partial y} \right)_{y=0} = -\frac{\lambda}{2} \sqrt{\frac{u_0}{\nu x}} \left(\frac{d\vartheta}{d\xi} \right)_{\xi=0}$$

$$= \frac{\lambda}{2} \sqrt{\frac{u_0}{\nu x}} \left[(T_w - T_0)\, a - \frac{u_0^2}{2\, g\, c_p}\, a\, b \right] \qquad (14)$$

Equations (13) and (14) can be transformed if the gas temperature in undisturbed flow is replaced by the temperature that the unheated wall assumes in the gas stream. This is henceforth termed "natural temperature" T_e. Since, in this instance, the wall neither absorbs nor diffuses heat, the temperature field is

$$\left(\frac{\partial \vartheta_2}{\partial y}\right)_{y=0} = 0; \quad \left(\frac{d \vartheta_2}{d\xi}\right)_{\xi=0} = 0$$

Performing this operation with equation (13) gives the temperature field at the unheated wall

$$\vartheta_2(\xi) = \frac{u_0^2}{2\ g\ c_p}\left(b - \frac{Pr}{2}\int_0^{\xi}\left[e^{-Pr\int_0^{\xi}\zeta d\xi}\int_0^{\xi}\left(\frac{d^2\zeta}{d\xi^2}\right)^2 e^{Pr\int_0^{\xi}\zeta d\xi}\, d\xi\right]d\xi\right) \tag{15}$$

where again*

$$b = \frac{Pr}{2}\int_0^{\infty}\left[e^{-Pr\int_0^{\xi}\zeta\ d\xi}\int_0^{\xi}\left(\frac{d^2\zeta}{d\xi^2}\right)^2 e^{Pr\int_0^{\xi}\zeta\ d\xi} d\xi\right] d\xi$$

The increase of temperature assumed by the wall follows from (15) for $\xi = 0$

$$\vartheta_e = \vartheta_2(0) = \frac{u_0^2}{2\ g\ c_p}\ b \tag{16}$$

With it the natural temperature of the wall itself is known: $T_e = T_0 + \vartheta_e$. The quantity $\dfrac{\vartheta_e}{u_0^2/2\ g\ c_p}$ can be read from figure 2.

In the range of Prandtl numbers $(Pr = 0.5 \text{ to } 2)$ involved for gases $\dfrac{\vartheta_e}{u_0^2/2\ g\ c_p} = \sqrt{Pr}$ is a very good approximate. From figure 3, wherein the adiabatic temperature rise $\dfrac{u_0^2}{2\ g\ c_p}$ is shown plotted against the speed u_0,

*This equation also agrees with the solution given by Pohlhausen for the plate thermometer.

the value itself can be formed without further calculation. Several experiments with air were made for predicting the natural temperature T_e. The recorded values are in good agreement with those to be marked off in figure 2. Figure 4 shows the temperature field in the boundary layer at an unheated wall for different Prandtl numbers. Since the temperature rise at the wall, according to figure 2, becomes so much greater for a given speed as the Prandtl number is greater, hence reaches especially great values for oils (in the lubricating film, for instance) (a case in point being that treated by G. Vogelpohl, in Öl u. Kohle, vol. 14, 1938, p. 996), equation (15) was evaluated up to $Pr = 1000$.

Subsequent to the introduction of T_e in equation (13) conformably to equation (16), the temperature field of the heated plate can be written in the form

$$\vartheta(\xi) = (T_w - T_e)\left[1 - a\int_0^\xi e^{-Pr\int_0^\xi \zeta\, d\xi}\, d\xi\right] +$$

$$+ \frac{u_o^2}{2\, g\, c_p}\left(b - \frac{Pr}{2}\int_0^\xi \left[e^{-Pr\int_0^\xi \zeta\, d\xi}\int_0^\xi \left(\frac{d^2\zeta}{d\xi^2}\right)^2 e^{Pr\int_0^\xi \zeta\, d\xi}\, d\xi\right] d\xi\right) \quad (17)$$

The first summand represents the temperature field $\vartheta_1(\xi)$ resulting at an increase of plate temperature $T_w - T_e$ in the absence of frictional heat; the second summand indicates the temperature rise $\vartheta_2(\xi)$ due to heat of friction. The temperature field therefore represents the superposition of the two separate fields, as is evident from the linear construction of equation 3. In figure 5 the temperature fields generated in this manner are shown for a Prandtl number of $Pr = 0.7$. Air, according to the most recent measurements (reference 7) has a Prandtl number of $Pr = 0.715$; the temperature fields reproduced in figure 5 are therefore sufficiently exact for this gas. The ordinate scale at the right, from which the increases of temperature for an airspeed of 200 meters per second can be read direct, affords a concept of the occurring speeds. The bracketed numerical values give the temperature differences $T_w - T_o$ for the same airspeed. The heat given off by the plate is equal to zero according to (14) and (16) for the line with the parameter $\frac{T_w - T_o}{\vartheta_e} = 1$, for the two temperature fields $\frac{T_w - T_o}{\vartheta_e} = 2$ and $\frac{T_w - T_o}{\vartheta_e} = 0$, equally great with the difference that in the second case the heat flow is directed toward the plate.

With the introduction of T_e the transfer of heat q from the wall to the gas (equation (14)) assumes the form

$$q = \frac{\lambda}{2}\sqrt{\frac{u_o}{\nu x}}\, a\ (T_w - T_e) \tag{18}$$

and the nondimensional local heat transfer coefficient, when referred, according to

$$q = \alpha(T_w - T_e) \tag{19}$$

to the difference between wall and natural temperature, follows at

$$Nu = \frac{\alpha\ x}{\lambda} = \frac{a}{2}\sqrt{Re} = 0.332\sqrt{Re}\,\sqrt[3]{Pr} \tag{20}$$

The Nusselt number formed with the mean heat transfer from beginning of plate to point x is

$$Nu_m = \frac{\alpha_m\ x}{\lambda} = \alpha\sqrt{Re} = 0.664\sqrt{Re}\,\sqrt[3]{Pr} \tag{21}$$

The heat transfer factors defined according to equation (19) therefore follow the same relations as at small flow velocities. The natural temperature required for calculating the heat flow q can be taken from figure 2.

Also of interest in many instancs is the amount of heat produced by internal friction. The frictional heat developed per unit volume is given by the term

$\nu\rho\left(\frac{\partial u}{\partial y}\right)^2$ of equation (3'). From it follows the heat

generated at point x in the total boundary layer by integration with respect to y at

$$q = \nu\rho\int_0^\infty\left(\frac{\partial u}{\partial y}\right)^2 dy = \frac{\nu\rho\, u_o^2}{8}\sqrt{\frac{u_o}{\nu x}}\int_0^\infty\left(\frac{d^2\zeta}{d\xi^2}\right)^2 d\xi \tag{22}$$

Evaluated, the integral $\int_0^\infty\left(\frac{d^2\zeta}{d\xi^2}\right)^2 d\xi$ yields 2.018. Now it is a simple matter to predict the temperature the wall must have in order that the total frictional heat flows onto the plate. It merely remains that equations (18) and (22) be equated. The resultant temperature field is shown as curve a in figure 5.

<u>Recheck of the derived formulas for $Pr = 1$.</u> Busemann and Von Kármán had computed, for a gas with a Prandtl number $Pr = 1$, the velocity profile with allowance for the variation of the material values. Now the reliability of the earlier assumption of constant material values for the calculation of the heat transfer is to be checked with the rsults of the present study. Between the velocity u in the boundary layer of a gas with a Prandtl number $Pr = 1$ and the absolute temperature T at the same point, there exists the relation (reference 2)

$$T = T_w - \left[T_w - T_o - \frac{u_o^2}{2\,g\,c_p}\right]\frac{u}{u_o} - \frac{u_o^2}{2\,g\,c_p}\left(\frac{u}{u_o}\right)^2 \qquad (23)$$

(T_w, plate temperature; T_o, u_o, temperature and speed in undisturbed gas flow outside the boundary layer). The unheated plate is characterized by the fact that the temperature gradient at the wall is equal to zero: $\partial T/\partial y = 0$. But, since the value of $\partial u/\partial y$ at the wall is certainly different from zero and the speed at the wall is zero, it implies that the bracketed term in equation (23) must become zero. The plate temperature in this instance—that is, the natural temperature — is therefore

$$T_e = T_o + \frac{u_o^2}{2\,g\,c_p}$$

in agreement with equation (16), where b assumes the value 1 for $Pr = 1$, according to figure 2. The temperature T_2 in the boundary layer of the unheated plate follows from equation (23):

$$T_2 = T_e - \frac{u_o^2}{2\,g\,c_p}\left(\frac{u}{u_o}\right)^2$$

The equation can also be written in the form

$$c_p\,T_2 + \frac{u^2}{2g} = c_p\,T_o + \frac{u_o^2}{2\,g}$$

where it expresses the well-known fact that for a gas with $Pr = 1$ in the boundary layer the sum of enthalpy and motion energy has the same value at every point of the unheated plate. This applies also, as is known, to the turbulent bounary layer (reference 8).

The ratio of increase of temperature in the boundary layer to that at the wall is

$$\frac{\vartheta_2}{\vartheta_e} = \frac{T_2 - T_0}{T_e - T_0} = 1 - \left(\frac{u}{u_0}\right)^2$$

The temperature field for $Pr = 1$, reproduced in figure 4, can therefore be derived in simple manner from the velocity profile (fig. 1).

After insertion of T_e in equation (23) the temperature field of the heated plate assumes the form

$$T = T_w - (T_w - T_e)\frac{u}{u_0} - \frac{u_0^2}{2\ g\ c_p}\left(\frac{u}{u_0}\right)^2 \tag{23a}$$

The temperature gradient at the wall follows at

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = (T_w - T_e)\ \frac{1}{u_0}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

because, since the speed at the wall is zero, the second term in the above equation disappears. Reference of the heat transfer factor again to the difference between wall temperature and natural temperature affords

$$\alpha = \frac{-\lambda_w}{T_w - T_e}\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\lambda_w\left[\frac{\partial\left(\frac{u}{u_0}\right)}{\partial y}\right]_{y=0} \tag{24}$$

λ and subsequently η, ν, and c_p are given the subscript w, since their values are to be entered for the wall temperature. Figure 6 shows the velocity profiles at the unheated plate for three Mach numbers plotted against the nondimensional distance from the plate $\xi = \frac{y}{2}\sqrt{\frac{u_0}{\nu_w\ x}}$ according to the calculations by Von Kármán and Tsien (reference 3). It is readily apparent that the discrepancies between the velocity profiles are small, especially at Mach numbers up to $M = 2$. The velocity gradient at the wall is only about 3 percent greater at $M = 2$ than at $M = 0$. Even at $M = 10$ the difference is no more than 20 percent. To the extent that the temperature differences $(T_w - T_e)$ do not become excessive at the heated plate, the velocity profiles do not vary perceptibly; hence the heat transfer factor which previously has been computed at all speeds with the velocity profile for $M = 0$ differs, at the most, by 3 percent from the true value of a gas at $Pr = 1$. At other Pr not too far from unity the error is of the same order of magnitude.

The foregoing arguments further indicate the temperature at which the material values should be entered in the heat transfer equations. Equation (24) already implied by subscript w that the thermal conductance must be introduced at the wall temperature. Even the differences in the velocity profiles at different M are much smaller, if, as in figure 6, the plate distance with the kinematic viscosity at the wall the temperature is made nondimensional. In figure 3 of the Von Kármán – Tsien report (reference 3) the velocity profiles are plotted against

$\xi = y\sqrt{\frac{u_o}{\nu_o x}}$ (ν_o, kinematic viscosity at temperature in undisturbed gas flow). The discrepancies between the individual profiles are substantially greater.

Reynolds' formula for the ratio of heat flow $Q = -\lambda_w(\partial T/\partial y)_{y=o}$ onto the plate to its towing resistance $W = \eta_w(\partial u/\partial y)_{y=o}$

$$\frac{Q}{W} = \frac{g\, c_p(T_o - T_w)}{u_o}$$

holds true at low speeds and for a material with the Prandtl number $Pr = 1$. At higher Mach numbers the ratio $\frac{Q}{W}$ follows from the foregoing at

$$\frac{Q}{W} = \frac{\lambda_w(T_e - T_w)}{\eta_w\, u_o}$$

or, with consideration to $Pr = g\, c_p\, \eta/\lambda = 1$

at $$\frac{Q}{W} = \frac{g\, c_{pw}(T_e - T_w)}{u_0}$$

Reynolds' formula therefore is equally applicable in gas dynamics when the natural wall temperature T_e is substituted in place of the gas temperature T_0. It applies to laminar as to turbulent boundary layer.

Turbulent boundary layer.- From the technical point of view the heat transfer in the turbulent boundayer layer is of much greater importance than in the laminar, since the latter changes to turbulent, even by completely undisturbed inflow, if the Reynolds number formed with the distance from the start of the plate reaches 500,000. This occurs a few centimeters from the plate at high speeds. An exact theoretical treatment of the flow and temperature conditions is in this instance not possible and feasible only under certain assumptions concerning the turbulent interchange. However, it can be stated that the linear superposition of the of the temperature fields, as it occurs in the laminar boundary layer, remains also in the turbulent so long as the material values inclusive of the density can be regarded as constant, which, as previously shown, is the case for the flat plate up to Mach numbers $M = 2$. This is manifested by the general differential equation for the heat flow in a mass particle

$$g\, \rho\, c_p \frac{D\vartheta}{dt} = \lambda \Delta^2 \vartheta + \eta \quad (\text{diss. fct. } (\underline{w})) \tag{25}$$

(D/dt = substantial differential quotient with respect to time, diss. fct. ($\underline{w}$) = dissipation function for the velocity vector $\underline{w}$). For, even this equation (25) is already linear in ϑ exactly as equation (3'), which results from the foregoing for steady flow with the possible omissions in the thin boundary layer. The field of flow is completely independent of the temperature field under the assumption ρ = constant, and therefore also the heat developed by internal friction: (η diss. fct. ($\underline{w}$)) a quantity defined by the velocity field.

The temperature field without these sources of heat is known from measurements at low speeds. The theory of similitude affords the aspect of the temperature field at high speeds but in absence of internal friction; namely, precisely as in a slow flow with the same value of Re and Pr, leaving for measurement then merely the temperature field formed at high airspeeds by internal friction at an unheated plate. And with it the field for heat transfer at high speed is obtained by superposition.

But from the linear superposition of the temperature fields, it further follows that in laminar as in turbulent boundary layer, the characteristic relation

$$Nu = f\ (Re,\ Pr) \tag{26}$$

established for the heat transfer at low speeds, must be equally applicable at high speeds, if the heat transfer factor is referred to the difference between the wall temperature and the natural temperature. Inasmuch as the heat transfer at low speeds is now fairly well established (reference 6) the knowledge of the "natural temperature" of the plate is sufficient for calculating the heat transfer at high speeds. According to calculations by Schirokow (reference 8) the natural temperature at the plate can be indicated. But it would be desirable to check these values by experiments.

Heat transfer on profiles.- If the body concerned is a profile with considerable thickness rather than a thin plate, the flow outside of the boundary layer is also affected by it. A velocity field and a pressure field is formed about the body. In this event the equation (2) of the boundary layer must be enlarged by the pressure gradient $-\partial p/\partial x$ on the right-hand side. Then the material values (density, viscosity, and the thermal conductance) vary on a surface normal within the boundary layer solely as a result of their temperature relationship. When this, as on the plate, is neglected for not too great Mach numbers, the boundary layer equations assume the form

$$\frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1"}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2"}$$

$$\lambda\frac{\partial^2 \vartheta}{\partial y^2} + \eta\left(\frac{\partial u}{\partial y}\right)^2 = g\ c_p \rho\ u\frac{\partial \vartheta}{\partial x} + v\frac{\partial \vartheta}{\partial y} \tag{3"}$$

where x denotes the coordinate along the surface of the body, y that along the surface normal, ϑ the increase of temperature over the value in the potential flow at the boundary of the boundary layer. The density gradient $\partial\rho/\partial x$ and the pressure gradient $\partial p/\partial x$, as well as the density ρ, the thermal conductance λ, and the viscosity η, are given as functions of x by the flow outside of the boundary layer. As a result, the velocity profile can be computed again simply from equations (1") and (2") and because of the linearity of equation (3"), the linear superposition of the temperature fields is applicable as on the plate. This implies that the local heat-transfer factor is independent of the temperature difference. Admittedly, the range of validity of the heat transfer formulas secured by tests at low velocities is smaller than at the plate even if the heat transfer factor is referred to the difference: wall temperature and natural temperature, since on approaching sonic velocity the field of flow outside of the boundary layer is slightly deformed owing to the variable density. On exceeding the sonic velocity it varies rather considerably due to the occurrence of compression shocks. With the knowledge of the external field of flow, the equations (1") to (3"), on the other hand, can be integrated and so the velocity and temperature field in the boundary layer solved. The solution for the flow in the neighborhood of a stagnation point is to be reported shortly.

For the cited reasons the application of the heat transfer factors on the flat plate to the heat exchange on airplane wings and fuselages, as is customary at low velocities, requires a certain caution. But, if the areas with especially great pressure differences (nose of wing) are excluded from the analysis, an approximate solution is certainly possible by means of the data for the flat plate. In conclusion, an estimate is given of the effect of the internal friction on the heat transfer at the initially mentioned applications in airplane design. On a water radiator mounted in the wing of an airplane and swept in flight with an air velocity of 200 meters per second, the temperature difference of the surface wetted by the water relative to air in flight in ground proximity, amounts to about 65°, the difference between natural temperature and air temperature, according to figures 2 and 3, to 17° C in the laminar boundary layer and between 17° and 18° C in the turbulent. The controlling difference for the heat transfer between wall temperature and natural temperature is therefore 47° to 48°; whereas without internal friction the total increase of 65° would be free for the heat transfer which would be, therefore, around 40 percent greater.

With glycol cooling the effect is about half as great, since corresponding to the higher temperature of glycol the increase of temperature of the wetted surface is about twice as great as with the water radiator. In the cabin of a stratosphere airplane without ventilation and without heating a temperature would result that is equal to the natural temperature of the surface of the airplane fuselage, hence at a flying speed of 200 meters per second is from 17° to 18° higher than the outside air temperature. The latter is -56.5° at flying heights above 11 kilometers. If the air temperature in the cabin is to be kept at +15° C by additional heating, it need only correspond to a 17° to 18° smaller temperature difference. The conditions are therefore similar to those obtaining on a wing radiator with water cooling, and at 200 meters per second flying speed the heat transmission through the insulation is reduced by about 40 percent as a result of the air friction. As the flying speed increases the effect of the internal friction increases very materially. From 380 meters per second airspeed on the water radiator ceases to give off heat at sea level, and insulation of the cabin of the stratosphere aircraft is no longer necessary. In lower flying levels provisions would already have to be made for cabin cooling.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Seibert, O.: Messungen über die Wärmeabgabe eines Profiles. Jahrb. 1938 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich) pp. II 224-244.

Seibert, O.: Heat Transfer of Airfoils and Plates. T.M. No. 1044, NACA, 1943.

2. Busemann, A.: Z. angew. Math. Mech. Bd. 15, 1935, p. 23.

3. Von Kármán, Th., and Tsien, H. S.: Boundary Layer in Compressible Fluids. Jour. Aer. Sci., vol. 5, no. 6, Apr. 1938, pp. 227-232.

4. Blasius, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Z. Math. Phys. Bd. 56, 1908, pp. 1-37.

5. Pohlhausen, E.: Der Wärmeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung. Z.f.a.M.M., Bd. 1, Heft 2, 1921, pp. 115-21.

6. Ten Bosch, M.: Die Wärmeübertragung. Julius Springer (Berlin), 1936, p. 136.

7. Henning, F.: Wärmetechnische Richtwerte. (Berlin) 1938.

8. Busemann, A.: In Wien-Harms, Handbuch der Experimentalphysik, Bd. IV, 1. Teil, (Leipzig) 1931, p. 365.

9. Schirokow, M.: Techn. Phys. USSR, Bd. 3, 1936, p. 1020.

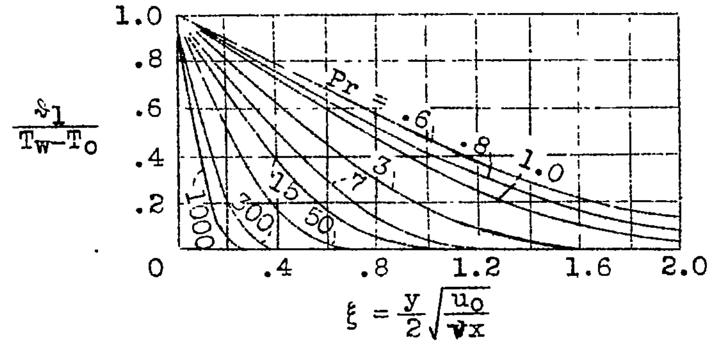

Figure 1.- Temperature field ϑ_1 on a heated plate in flow at low speed for different Prandtl numbers Pr in laminar boundary layer. T_W = wall temperature, T_o = temperature in undisturbed gas flow.

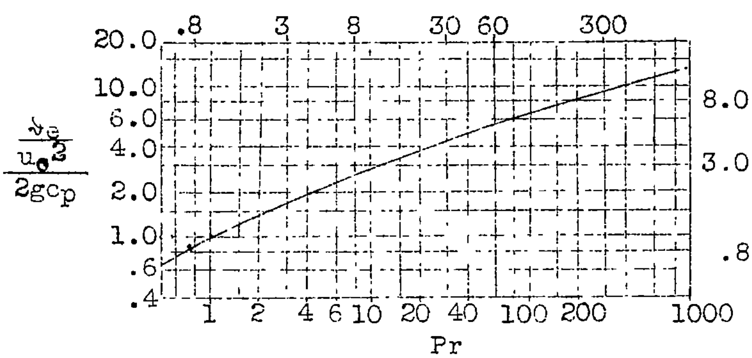

Figure 2.- Increase of temperature ϑ_e of an unheated plate in flow at high speed (u_o) plotted against Pr in laminar boundary layer.

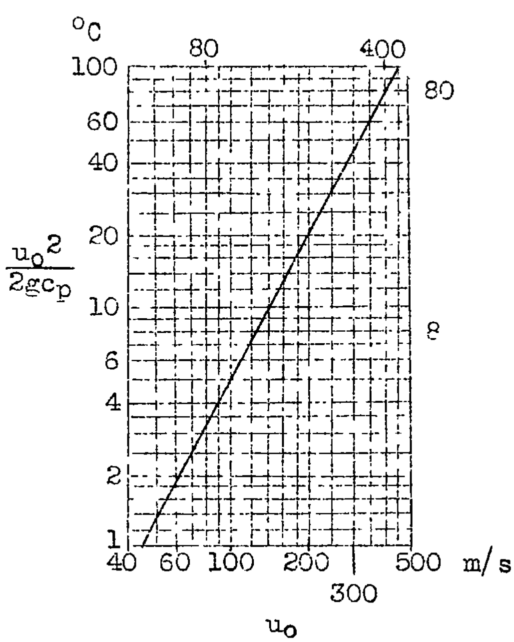

Figure 3.- Temperature rise introduced by adiabatic concentration of a flowing gas with velocity (u_o) to velocity zero.

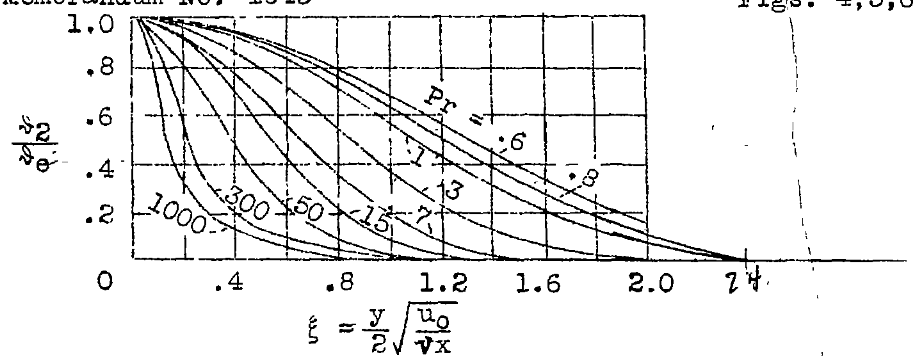

Figure 4.- Temperature field ϑ_2 of an unheated plate in flow at high speed at different Pr in laminar boundary layer.

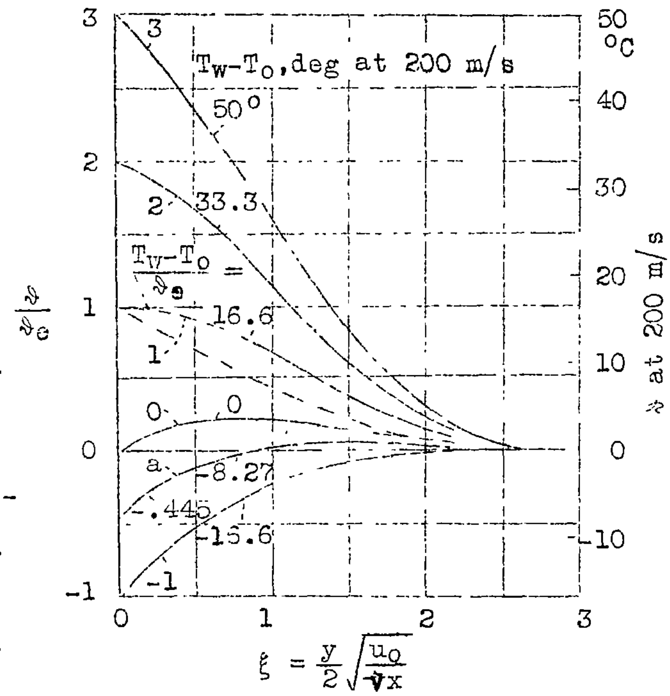

Figure 5.- Temperature field ϑ on a heated plate in flow at high speed of Pr = 0.7 with laminar boundary layer in the temperature field, curve a, the total heat produced in the boundary layer by internal friction is transferred to the plate.

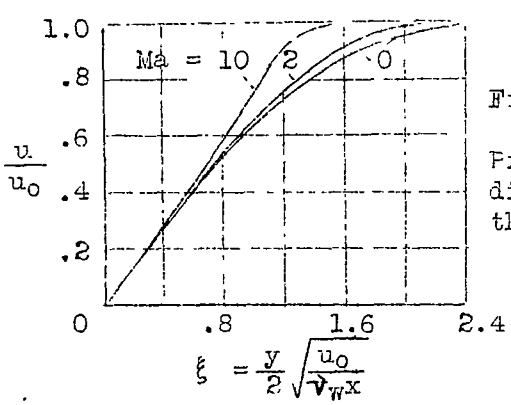

Figure 6.- Velocity field (u/u_0) in the boundary layer of a gas with Pr = 1 flowing past a flat plate at different Mach numbers plotted against the nondimensional plate distance ξ.